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Saddle-Point Theorems for Rational Approximation*

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Rational approximation is considered where the $(n+m)$ functions involved are supposed to be continuous on a general compact set. With the aid of Helly-type theorems, the approximation viewed as a mathematical program is reduced to one that is discrete, without any assumption regarding the existence of solutions. This discrete problem, which is the rational approximation considered on an at most $(n+m)$ element subset of our compact set, has the same value as the original problem, while its solution set includes that of the original problem. Moreover, all the above sets of cardinality at most $(n+m)$ are found by max–inf statements, where the maximum interchange with the infimum and a finite number of variables are involved. If the original approximation problem has a solution, then all of its solutions, as well as all the above-mentioned finite subsets, are expressed by the saddle points of our minimax statements.

1. INTRODUCTION

The problem of rational approximation on a compact set T is defined as follows: $P_1,..., P_n, Q_1,..., Q_m$, and f being real continuous functions defined on T, solve

$$\min \left\{ \left\| f - \frac{x_1 P_1 + \cdots + x_n P_n}{y_1 Q_1 + \cdots + y_m Q_m} \right\|_\infty \;\middle|\; (x, y) \in R^n \times R^m, \right.$$
$$\left. \forall \theta \in T\colon \sum_{k=1}^{m} y_k Q_k(\theta) > 0 \right\}, \quad (1)$$

where $\|\cdot\|_\infty$ denotes the Chebyshev norm.

Since we are dealing with rational approximation on compact sets $T' \subseteq T$,

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and the problem is being viewed as a mathematical program, we consider in this paper the following semi-infinite program $\mathbf{P}(T')$:

$$\inf \mathbf{P}(T') \triangleq \inf_{x,y} \left\{ \max_{t\in T'} \left| f(t) - \frac{x^T P(t)}{y^T Q(t)} \right| \;\middle|\; \forall \theta \in T' \colon y^T Q(\theta) > 0 \right\} \quad \mathbf{P}(T'),$$

where $x \triangleq (x_1,\dots,x_n)^T$, $y \triangleq (y_1,\dots,y_m)^T$, $P(t) \triangleq (P_1(t),\dots,P_n(t))^T$, and $Q(t) \triangleq (Q_1(t),\dots,Q_m(t))^T$.

In the case where $T' = T$, clearly $\mathbf{P}(T')$ is equivalent to (1). When T' is a finite subset of T, the discretized approximation program $\mathbf{P}(T')$, is here called a "reduced program," as this term probably accords better with the mathematical programming concept. Note that the constraints involved in $\mathbf{P}(T')$ are expressed in terms of T' only, and thus the problem $\mathbf{P}(T')$ is an independent program. In this paper we do not assume the existence of solutions of $\mathbf{P}(T')$ for any $T' \subseteq T$; also we assume no properties of f, P, Q, and T other than continuity and sequential compactness.

We have two aims. Our first is to apply Helly-type theorems in order to make a certain reduction (discretization) by finding a subset T_0 of T containing no more than $(n+m)$ points such that $\inf \mathbf{P}(T_0) = \inf \mathbf{P}(T)$. This equality implies that any minimizing sequence of $\mathbf{P}(T)$ is a minimizing sequence of $\mathbf{P}(T_0)$. Our second aim is to express all the sets T_0 satisfying the above-mentioned property by max–inf statements, where the maximum interchange with the infimum and a finite number of variables are involved. One of these statements involves a Lagrangian having a certain differentiable property. In the case where $\mathbf{P}(T)$ has solutions, all of them as well as all the above T_0 sets are expressed by the saddle points of our minimax statements.

The existence of a subset $T_0 \subseteq T$ having at most $(n+m)$ elements and satisfying $\inf \mathbf{P}(T) = \inf \mathbf{P}(T_0)$ is known only in the case where the original approximation problem $\mathbf{P}(T)$ has a solution (see [4, 8, 9]). To the best of the writer's knowledge, the only result of this type for the case where the original problem $\mathbf{P}(T)$ has no solution is that of Krabs (see [8, 9]) who proved the existence of a subset $T_0 \subseteq T$ having at most $(n+m+1)$ elements and satisfying $\inf \mathbf{P}(T) = \inf \mathbf{P}(T_0)$. In Section 2 we improve Krabs' result in the sense that we delete one more element from the above T_0 subsets. As a second aspect we use a purely geometrical technique, which is general and differs from the Kolmogorov principle. Our development is essentially based on a Helly-type theorem related to the intersection of an infinite number of convex sets. This theorem, due to Klee, uses the concept of 0-closeness (see [2, 7]).

DEFINITION. A family Γ of sets is called 0-closed if every set in Γ is open and $\operatorname{int} K \in \Gamma$ whenever K is the limit of a convergent sequence of sets in Γ. Here the limit of $K_n \in \Gamma$ is $\lim_{n\to\infty} K_n = \bigcup_{1\leq i} \bigcap_{i\leq n} K_n = \bigcap_{1\leq i} \bigcup_{i\leq n} K_n$.

KLEE'S THEOREM. *Let Γ be an 0-closed family of convex sets in R^n. Then the intersection of all members of Γ is empty iff there are at most $(n+1)$ members of Γ the intersection of which is empty.*

The structure of this paper is as follows. In Section 2 we reduce our original program $\mathbf{P}(T)$ to $\mathbf{P}(T_0)$ having the property already mentioned. In Section 3 we present two saddle-point theorems expressing all the minimizing sequences of $\mathbf{P}(T)$ and all $(n+m)$-element sets T_0 by inf–max statements, where the infimum and the maximum are interchangeable. In the case where the original approximation problem $\mathbf{P}(T)$ has a solution, all its solutions and all the above T_0 subsets of T are expressed by saddle-point statements. One of the above theorems uses the Lagrangian presented in theorems and examples appearing in Section 3 of [6]. The reason why this Lagrangian is used, is that it has the property of being differentiable (in certain cases) on an open set containing all its saddle points.

2. THE REDUCTION TO THE FINITE CASE

We begin this section with a lemma which, together with its proof, is similar to Lemma 2.3 of [2] but is based on slightly modified assumptions that necessitate certain modifications of the original proof. We also use the same notation as in [2]. The theorem making it possible to reduce our semi-infinite program $\mathbf{P}(T)$ to the usual one involving $(n+m)$ functions, is proved in two steps. First, with the aid of the above-mentioned lemma we find an $(n+m+1)$-element subset of T satisfying the desired property. Finally, using a Helly-type theorem for cones, we delete one more element and obtain the desired set T_0.

Krabs' result could be used as the first step of our reduction theorem. However, we prefer to use a modified version of the Ben-Israel *et al.* lemma [2] for the following reasons:

(a) the argument is more general and its range of applicability includes (semi-infinite) quasi-convex programs. Note that the original lemma of Ben-Israel *et al.* has been successfully applied for reducing semi-infinite convex programs to ordinary ones. Thus, this argument is applicable to a wider class of approximation problems;

(b) the procedure illustrates a unique technique, which is essentially the application of Helly-type theorems to reduce (discretize) a Chebyshev approximation problem.

LEMMA 1. *Let $\alpha \in R$ and let T be a compact set. Also let $g(x, t)$: $R^n \times T \to [-\infty, \infty]$ be a given function satisfying the following properties:*

(A1) *for all fixed* $x \in R^n$, $g(x, \cdot)$ *is upper semicontinuous on* T;

(A2) *for all fixed* $t \in T$:

$$K(t) \triangleq \{x \mid g(x, t) < \alpha, x \in R^n\}$$

is an open convex set and

$$H(t) \triangleq \{x \mid g(x, t) = \alpha, x \in R^n\}$$

has the property that int $\bar{H}(t) = \varnothing$ *if* $K(t) \neq \varnothing$, *the superscript* $^-$ *indicating the topological closure*;

(A3) *for any nonvoid open set* $\mathcal{O} \in R^n$ *and any fixed* $t \in T$: *if* $g(\cdot, t) > \alpha$ *on* $\mathcal{O}$, *there are an* $x_0 \in \mathcal{O}$ *and a vicinity* $\mathcal{V}$ *of* t *such that* $g(x_0, \cdot) > \alpha$ *on* $\mathcal{V}$.

Then $\bigcap_{t \in T} K(t) = \varnothing$ *implies the existence of* $t_1, \dots, t_k \in T$ *such that* $k \leqslant n+1$ *and* $\bigcap_{i=1}^{k} K(t_i) = \varnothing$.

Proof. If there exists $t \in T$ such that $K(t) = \varnothing$, the assertion follows trivially. Therefore, we consider the second case, which is $K(t) \neq \varnothing$ for any $t \in T$, and thus, by Klee's theorem, it is sufficient to prove that the family

$$\Gamma \triangleq \{K(t) \mid t \in T\}$$

is 0-closed. Let $K(t_n)$ be any converging sequence and denote $K = \lim_{n\to\infty} K(t_n)$. We shall prove that int $K \in \Gamma$ (i.e., $\exists t^* \in T \ni \operatorname{int} K = K(t^*)$). Let t_m be a subsequence of t_n such that $\lim_{m\to\infty} t_m = t^* \in T$. Then

$$\begin{aligned} \bigcap_{1 \leqslant i} \bigcup_{i \leqslant m} K(t_m) &\subset \bigcap_{1 \leqslant i} \bigcup_{i \leqslant n} K(t_n), \\ \bigcup_{1 \leqslant i} \bigcap_{i \leqslant n} K(t_n) &\subset \bigcup_{1 \leqslant i} \bigcap_{i \leqslant m} K(t_m). \end{aligned} \tag{2}$$

This means that $\lim_{m\to\infty} K(t_m) = K$. Moreover, for any $x \in K(t^*)$ we have $g(x, t^*) < \alpha$, and thus by (A1), for any m that is large enough, $g(x, t_m) < \alpha$, i.e., $x \in K(t_m)$. Using (2) we obtain $K(t^*) \subset K$ and thus,

$$K(t^*) \subset \operatorname{int} K. \tag{3}$$

Furthermore, if $K(t^*) \neq \operatorname{int} K$, then by (3) and the fact that $K(t^*)$ is a convex set,

$$\operatorname{int}[\operatorname{int} K \backslash K(t^*)] \neq \varnothing,$$

(see [2]). Applying (A2) we get

$$\varnothing \neq \operatorname{int}[\operatorname{int} K \backslash K(t^*)] \backslash \overline{H(t^*)} \subset \operatorname{int}[\operatorname{int} K \backslash (K(t^*) \cup \overline{H(t^*)})],$$

which implies the existence of an open set $\mathscr{O} \subset \operatorname{int} K$ such that $g(\cdot, t^*) > \alpha$ on $\mathscr{O}$. Therefore by (A3), there are an $x_0 \in \mathscr{O}$ and a vicinity $\mathscr{V}$ of t^* such that $g(x_0, \cdot) > \alpha$ on $\mathscr{V}$. It follows that for any m that is large enough,

$$g(x_0, t_m) > \alpha,$$

which implies that $x_0 \notin K$. This contradicts $x_0 \in \mathscr{O} \subset K$. Thus, $K(t^*) = \operatorname{int} K$, implying that Γ is an 0-closed family; which completes the proof. ∎

LEMMA 2. *Let $a_i \in R^n$, $i = 1,..., p$ $(n + 2 \leqslant p)$ such that for some $z \in R^n$, $z^T a_i > 0$ for each $i = 1,..., p$. We denote by $\mathscr{K}_j$ the following sets:*

$$\mathscr{K}_j \triangleq \bigcap_{\substack{i=1 \\ i \neq j}}^{p} \{y \mid y^T a_i > 0\}.$$

Then $0 \notin \mathscr{K}_1 + \cdots + \mathscr{K}_p$.

Proof. For any set $\mathscr{X}$ we shall denote by $\operatorname{conv} \mathscr{X}$ and $\operatorname{co} \mathscr{X}$, respectively, the convex hull and the cone generated by the set $\mathscr{X}$. Any $(p-1)$ sets belonging to the family

$$\left\{ \operatorname{conv} \bigcup_{\substack{i=1 \\ i \neq j}}^{p} \{a_i\} \right\}_{j=1,...,p}$$

have a joint point. Indeed, for any fixed $j_0 \in \{1,..., p\}$ we have

$$\forall j \in \{1,..., p\} \backslash \{j_0\} \colon a_{j_0} \in \operatorname{conv} \bigcup_{\substack{i=1 \\ i \neq j}}^{p} \{a_i\}.$$

Thus by Helly's theorem there is a $y \in R^n$ such that

$$y \in \bigcap_{j=1}^{p} \operatorname{conv} \bigcup_{\substack{i=1 \\ i \neq j}}^{p} \{a_i\} \subset \bigcap_{j=1}^{p} \operatorname{co} \bigcup_{\substack{i=1 \\ i \neq j}}^{p} \{a_i\}.$$

Moreover, by our assumption on $\{a_i \mid i = 1,..., p\}$, $y \neq 0$. Therefore, denoting by $\bar{\mathscr{K}}_j$, $j = 1,..., p$ the closed cones

$$\bar{\mathscr{K}}_j \triangleq \bigcap_{\substack{i=1 \\ i \neq j}}^{p} \{y \mid y^T a_i \geqslant 0\},$$

we obtain that

$$R^n \neq \left[\bigcap_{j=1}^{p} \operatorname{co} \bigcup_{\substack{i=1 \\ i \neq j}}^{p} \{a_i\} \right]^* = \sum_{j=1}^{p} \left[\operatorname{co} \bigcup_{\substack{i=1 \\ i \neq j}}^{p} \{a_i\} \right]^* = \sum_{j=1}^{p} \bar{\mathscr{K}}_j \triangleq \mathscr{K}$$

(see [1]), where the superscript * denotes the polar cones. Thus $0 \notin \operatorname{int} \mathcal{K}$ and, since $\mathcal{K}_j'$, $j = 1,\dots, p$, are open sets,

$$\mathcal{K}_1 + \cdots + \mathcal{K}_p \subset \operatorname{int} \mathcal{K},$$

which completes the proof. ∎

Lemma 3. *Let $\mathcal{K}_i \subset R^n$, $i = 1,\dots, p$ $(n+1 \leqslant p)$ be cones, and suppose that for each $j = 1,\dots, p$ there are $x_j \in R^n$ satisfying the conditions:*

(a) $x_j \in \bigcap_{i=1, i \neq j}^{p} \mathcal{K}_i$;

(b) $0 \notin \operatorname{conv} \bigcup_{j=1}^{p} \{x_j\}$.

Then $\bigcap_{i=1}^{p} \mathcal{K}_i \neq \{0\}$.

Proof. Denote $x_0 \triangleq 0$. Using Radon's theorem we obtain that there is a partition $\{I; J\}$ of $\{0, 1,\dots, p\}$ such that

$$y \in \left[\operatorname{conv} \bigcup_{i \in I} \{x_i\}\right] \cap \left[\operatorname{conv} \bigcup_{i \in J} \{x_i\}\right] \subset \bigcap_{i \in J} \mathcal{K}_i \cap \bigcap_{i \in I} \mathcal{K}_i.$$

To complete the proof, we remark only that by assumption (b), $y \neq 0$. ∎

For any natural number q we define the following family Σ^q of sets:

$$\Sigma^q \triangleq \{T_0 \mid T_0 \subseteq T, \operatorname{card} T_0 \leqslant q, \inf \mathbf{P}(T_0) = \inf \mathbf{P}(T)\}.$$

Theorem 1 (The reduction theorem). *Assume that we have a compact set T, a continuous function $f\colon T \to R$, and two continuous vector functions $P\colon T \to R^n$ and $Q\colon T \to R^m$. Suppose that there exists $y \in R^m$ such that $y^T Q(t) > 0$ for any $t \in T$. Then*

(i) *$\Sigma^{n+m} \neq \varnothing$ (or equivalently there is a finite set $T_0 \subseteq T$ containing at most $(n+m)$ distinct points such that* $\inf \mathbf{P}(T_0) = \inf \mathbf{P}(T)$);

(ii) *each subset $T_0 \subseteq T$ satisfying* $\inf \mathbf{P}(T_0) = \inf \mathbf{P}(T)$ *has the property that any minimizing sequence of* $\mathbf{P}(T)$ *is a minimizing sequence of* $\mathbf{P}(T_0)$.

Proof. Define the following function $g(x, y; t)\colon R^n \times R^m \times T \to [0, \infty]$:

$$g(x, y; t) \triangleq \left| f(t) - \frac{x^T P(t)}{y^T Q(t)} \right|, \qquad y^T Q(t) > 0,$$
$$\triangleq \infty, \qquad \text{otherwise}. \tag{4}$$

Note that, by our assumption, for any subset $T' \subseteq T$ we have $0 \leqslant \inf \mathbf{P}(T') < \infty$ and our program $\mathbf{P}(T')$ is exactly

$$\inf \mathbf{P}(T') = \inf_{x, y} \max_{t \in T'} g(x, y; t). \tag{5}$$

The property (A1) as well as the first part of (A2) are trivially satisfied (where (x, y) is considered instead of the variable x and $\alpha \triangleq \inf \mathbf{P}(T)$). For the second part of (A2) we remark that if for some $t \in T$ and two open sets $\mathcal{O}_1 \subset R^n$ and $\mathcal{O}_2 \subset R^m$,

$$\forall (x, y) \in \mathcal{O}_1 \times \mathcal{O}_2 \colon g(x, y; t) = \inf \mathbf{P}(T),$$

then $P(t) = 0$ and $|f(t)| = \mathbf{P}(T)$, which implies that

$$\{(x, y) \mid g(x, y; t) < \inf \mathbf{P}(T)\} = \varnothing.$$

Remarking that the set

$$\{(x, y) \mid g(x, y; t) = \inf \mathbf{P}(T)\}$$

and its closure have the same interior, the second part of (A2) is satisfied.

In order to show that (A3) holds, we suppose that $t_0 \in T$ is given, together with two nonvoid open sets $\mathcal{O}_1 \subset R^n$ and $\mathcal{O}_2 \subset R^m$ such that

$$\forall (x, y) \in \mathcal{O}_1 \times \mathcal{O}_2 \colon g(x, y; t_0) > \mathbf{P}(T). \tag{6}$$

By our assumption $Q(t_0) \neq 0$, which implies that $\{ y^T Q(t_0) \mid y \in \mathcal{O}_2\}$ is an open set in R. Therefore, there is a $y_0 \in \mathcal{O}_2$ such that

$$y_0^T Q(t_0) < 0 \qquad \text{or} \qquad y_0^T Q(t_0) > 0.$$

Therefore, by the continuity property of $Q(\cdot)$ there is a vicinity $\mathcal{V}$ of t_0 such that one of the following statements holds:

$$\{\forall t \in \mathcal{V} \colon y_0^T Q(t) < 0\}; \qquad \{\forall t \in \mathcal{V} \colon y_0^T Q(t) > 0\}. \tag{7}$$

Let $x_0 \in \mathcal{O}_1$. If the first statement of (7) holds, the definition of $g(x, y; t)$ clearly yields

$$\forall t \in \mathcal{V} \colon g(x_0, y_0; t) = \infty > \inf \mathbf{P}(T).$$

If the second statement of (7) holds, the continuity of $g(x_0, y_0; \cdot)$ on $\mathcal{V}$, together with (6), yields (A3).

Therefore, applying Lemma 1 we obtain the existence of $t_1, \ldots, t_k \in T$ with $k \leqslant n + m + 1$ such that

$$\bigcap_{i=1}^{k} \{(x, y) \mid g(x, y; t_i) < \inf \mathbf{P}(T)\} = \varnothing. \tag{8}$$

In the case where $k = n + m + 1$ we prove the following *assertion*: there is a $j_0 \in \{1,..., n + m + 1\}$ such that

$$\bigcap_{\substack{i=1 \\ i \neq j_0}}^{n+m+1} \{(x, y) \mid g(x, y; t_i) < \inf \mathbf{P}(T)\} = \varnothing.$$

Suppose this assertion to be false. Then for each $j \in \{1,..., n + m + 1\}$ there is an $(x_j, y_j) \in R^n \times R^m$ such that

$$(x_j, y_j) \in \bigcap_{\substack{i=1 \\ i \neq j}}^{n+m+1} \{(x, y) \mid g(x, y; t_i) < \inf \mathbf{P}(T)\}.$$

These sets are cones without origin, $n \geqslant 1$, and for each $j = 1,..., n + m + 1$,

$$y_j \in \bigcap_{\substack{i=1 \\ i \neq j}}^{n+m+1} \{y \mid y^T Q(t_i) > 0\}.$$

Therefore, Lemmas 2 and 3 imply

$$\bigcap_{i=1}^{n+m+1} \{(x, y) \mid g(x, y; t_i) < \inf P(T)\} \neq \varnothing$$

which contradicts (8). Thus the above assertion having been proved, and denoting

$$\begin{aligned} T_0 &\triangleq \{t_1,..., t_{n+m+1}\} \backslash \{j_0\} \qquad &&\text{if} \quad k = n + m + 1, \\ &\triangleq \{t_1,..., t_k\} &&\text{if} \quad k < n + m + 1, \end{aligned}$$

it follows that

$$\inf \mathbf{P}(T) \leqslant \inf \mathbf{P}(T_0).$$

The converse inequality is obtained from

$$\forall (x, y) \in R^n \times R^m: \max_{t \in T_0} g(x, y; t) \leqslant \max_{t \in T} g(x, y; t),$$

by taking the infimum by (x, y) on both sides, and also using (5). This completes the proof of (i). Remarking that the last inequality holds for any $T_0 \subseteq T$, the equality $\inf \mathbf{P}(T) = \inf \mathbf{P}(T_0)$ implies part (ii). ∎

3. Saddle-Point Theorems

In this section we express by inf–max statements all the minimizing sequences of $\mathbf{P}(T)$ and all the sets T_0 given by the reduction theorem (Theorem 1). A particular case is derived when $Q(\cdot) \equiv 1 \in R$. First we define the function $\tilde{g}(x, y; t)$ as

$$\tilde{g}(x, y; t) = \left| f(t) - \frac{x^T P(t)}{y^T Q(t)} \right|, \qquad y^T Q(t) \geqslant 1,$$
$$= \infty, \qquad \text{otherwise,}$$

and for any compact $T' \subseteq T$, we consider the following program $\tilde{\mathbf{P}}(T')$:

$$\inf \tilde{\mathbf{P}}(T') \triangleq \inf_{x,y} \max_{t \in T'} \tilde{g}(x, y; t).$$

Remark. Since T' is a compact set and $Q(t)$ is a continuous vector function, then for any fixed $y \in R^m$ satisfying $y^T Q(t) > 0$ for each $t \in T'$, there exists $\lambda = \lambda(y) > 0$ such that $y^T Q(t) \geqslant 1/\lambda$ for each $t \in T'$. Therefore, the following statement is easily seen to be valid.

For any compact set $T' \subseteq T$: $\inf \mathbf{P}(T') = \inf \tilde{\mathbf{P}}(T')$, any minimizing sequence of $\tilde{\mathbf{P}}(T')$ is a minimizing sequence of $\mathbf{P}(T')$, and for any minimizing sequence (x_l, y_l) of $\mathbf{P}(T)$ there are $\Lambda_l = \Lambda(y_l) > 0$ such that for any $\mu_l \geqslant \Lambda_l$ $l = 1, 2, \ldots$, $\mu_l(x_l, y_l)$ is a minimizing sequence of $\tilde{\mathbf{P}}(T')$. ∎

From this point of view, for any compact set $T' \subseteq T$, the programs $\mathbf{P}(T')$ and $\tilde{\mathbf{P}}(T')$ are equivalent. We shall present two saddle-point theorems for $\tilde{\mathbf{P}}(T)$.

Notation. Denote $\omega \triangleq \min\{q \mid \Sigma^q \neq \varnothing\} = \min\{\operatorname{card} T_0 \mid T_0 \in \Sigma^{n+m}\}$. Also for any $\alpha_1, \ldots, \alpha_q \in R$, denote

$$\bigvee_{i=1}^{q} \alpha_i \triangleq \max\{\alpha_1, \ldots, \alpha_q\}.$$

Proposition 3. *Assume, given a compact set T, a continuous function $f: T \to R$, and two continuous vector functions $P: T \to R^n$ and $Q: T \to R^m$. Suppose that there is a $y \in R^m$ such that $y^T Q(t) > 0$ for each $t \in T$. Then for any integer q with $\omega \leqslant q \leqslant n + m$:*

(i) $\inf \mathbf{P}(T) = \inf_{x,y} \max_{t_i \in T, i=1,\ldots,q} \bigvee_{i=1}^{q} \tilde{g}(x, y; t_i) = \max_{t_i \in T, i=1,\ldots,q} \inf_{x,y} \bigvee_{i=1}^{q} \tilde{g}(x, y; t_i)$; (9)

(ii) *$(t_1, \ldots, t_q)$ is a solution of the maximization problem appearing in (9) iff $\bigcup_{i=1}^{q} \{t_i\} \in \Sigma^q$; (x_l, y_l) is a minimizing sequence of the infimum problem appearing in (9) iff (x_l, y_l) is a minimizing sequence of $\tilde{\mathbf{P}}(T)$;*

(iii) $[x, y; t_1, \ldots, t_q]$ *is a saddle point of* (9) *iff* (x, y) *is a solution of* $\tilde{\mathbf{P}}(T)$ *and* $\bigcup_{i=1}^{q}\{t_i\} \in \Sigma^q$.

Proof. Clearly, by our remark,

$$\inf \mathbf{P}(T) = \inf_{x,y} \max_{t \in T} \tilde{g}(x, y; t) = \inf_{x,y} \max_{\substack{t_i \in T \\ i=1,\ldots,q}} \bigvee_{i=1}^{q} \tilde{g}(x, y; t_i)$$

$$\geqslant \max_{\substack{t_i \in T \\ i=1,\ldots,q}} \inf_{x,y} \bigvee_{i=1}^{q} \tilde{g}(x, y; t_i).$$

Thus, for any $t_1, \ldots, t_q \in T$, our remark with $T' \triangleq \{t_1, \ldots, t_q\}$ implies

$$\inf \mathbf{P}(T) \geqslant \inf_{x,y} \bigvee_{i=1}^{q} \tilde{g}(x, y; t_i) = \inf_{x,y} \bigvee_{i=1}^{q} g(x, y; t_i). \tag{10}$$

Moreover, we obtain equality in (10) exactly for those sets $T_0 = \{t_1, \ldots, t_q\} \in \Sigma^q$. The proof is completed by invoking the statement of our remark with $T' \triangleq T$. ∎

Let $t_1, \ldots, t_q \in T$. We have

$$\alpha^* = \alpha^*(t_1, \ldots, t_q) \triangleq \inf_{x,y} \bigvee_{i=1}^{q} \tilde{g}(x, y; t_i)$$

$$= \left\{ \alpha \;\middle|\; \begin{array}{l} |x^T P(t_i) - y^T Q(t_i) f(t_i)| \leqslant \alpha y^T Q(t_i), \\ y^T Q(t_i) \geqslant 1, \qquad (x, y) \in R^n \times R^m, \forall i = 1, \ldots, q, \end{array} \right\}$$

or, equivalently, for any fixed $0 < \eta < 1$ and using the notation $\beta \triangleq 1/(1+\alpha)$

$$\frac{1}{1+\alpha^*} = \inf\{\beta \mid \beta||Ax - DBy| + \eta \vee By| \leqslant By, By \geqslant 1\}, \tag{11}$$

where A is $q \times n$ matrix, the rows of which are $P^T(t_i)$, $i = 1, \ldots, q$; B is the $q \times m$ matrix, the rows of which are $Q^T(t_i)$, $i = 1, \ldots, q$; and D is the $q \times q$ diagonal matrix, the diagonal of which is $f(t_i)$, $i = 1, \ldots, q$. We also denote for any vector $z \in R^q$,

$$|z| \triangleq \begin{bmatrix} |z_1| \\ \vdots \\ |z_q| \end{bmatrix}; \qquad \eta \vee z \triangleq \begin{bmatrix} \eta \vee z_1 \\ \vdots \\ \eta \vee z_q \end{bmatrix},$$

(see [6]). For any fixed $0 < \eta < 1$ we define a Lagrangian (tacitly depending on η), where $\tau \triangleq (t_1, \ldots, t_q) \in T \times \cdots \times T \triangleq T^q$,

$$L^q = L^q(x, y; u, s, \tau) \triangleq \frac{u^T(By - 1) + s^T By}{s^T[|Ax - DBy| + \eta \vee By]}. \tag{12}$$

In our remaining development we shall use the notation

$$\Delta^q \triangleq \left\{ s \;\middle|\; s \in R^q_+, \sum_{i=1}^{q} s_i = 1 \right\}, \qquad \Pi^q \triangleq R^q_+ \times \Delta^q.$$

THEOREM 3. *Assume given a compact set T, a continuous function $f: T \to R$, and two continuous vector functions $P: T \to R^n$ and $Q: T \to R^m$. Suppose that there is a $y \in R^m$ such that $y^T Q(t) > 0$ for each $t \in T$. Then for any integer q with $\omega \leqslant q \leqslant n + m$:*

(i) $1/(1 + \inf \mathbf{P}(T)) = \sup_{x,y} \min_{(u,s) \in \Pi^q, \tau \in T^q} L^q(x, y; u, s, \tau) = \min_{(u,s) \in \Pi^q, \tau \in T^q} \sup_{x,y} L^q(x, y; u, s, \tau)$;

(ii) $\tau = (t_1, \ldots, t_q)$ *is a solution of the minimization problem appearing in* (i) *iff* $\bigcup_{i=1}^q \{t_i\} \in \Sigma^q$; (x_l, y_l) *is a maximizing sequence of the supremum in* (i) *iff* (x_l, y_l) *is a minimizing sequence of* $\tilde{\mathbf{P}}(T)$;

(iii) *if* $[x, y; u, s, \tau]$ *with* $\tau = (t_1, \ldots, t_q)$ *is a saddle point of* L^q, *then* (x, y) *is a solution of* $\tilde{\mathbf{P}}(T)$ *and* $\bigcup_{i=1}^q \{t_i\} \in \Sigma^q$; *if* (x, y) *is a solution of* $\tilde{\mathbf{P}}(T)$ *and* $T_0 \in \Sigma^q$ *then there are* $s \in \Delta^q$ *and* $u \in R^q_+$ *such that* $[x, y; u, s, \tau]$ (*where the components of* τ *are the points of* T_0) *is a saddle point of* L^q.

Proof. For any fixed $(x, y) \in R^n \times R^m$ and $\tau \in T^q$:

(a) if $By \not\geqslant 1$, then

$$\min_{(u,x) \in \Pi^q} L^q(x, y; u, s, \tau) = -\infty,$$

since the denominator in (12) is always positive;

(b) if $By \geqslant 1$, then $s^T By > 0$ and

$$\min_{(u,x) \in \Pi^q} L^q = \min_{s \in \Delta^q} \frac{s^T By}{s^T[|Ax - DBy| + By]}$$

$$= 1 \Big/ \left(1 + \max_{s \in \Delta^q} \frac{s^T |Ax - DBy|}{s^T By} \right) = 1 \Big/ \left(1 + \bigvee_{i=1}^{n+m} \tilde{g}(x, y; t_i) \right).$$

Thus, taking the minimum by $\tau \in T^q$ and supremum by $(x, y) \in R^n \times R^m$, Proposition 3 implies

$$\frac{1}{1 + \inf \mathbf{P}(T)} = \sup \min L^q,$$

and its maximizing sequences are exactly the minimizing sequences of $\inf \tilde{\mathbf{P}}(T)$.

Now let τ be a fixed point in T^q, and consider the program (11) (which as has been shown is equivalent to that appearing in (10)). Applying the particular case 1 or the example, both appearing in Section 3 of [6], we obtain

$$\frac{1}{1+\alpha^*} = \min_{(u,s)\in\Pi^q} \sup_{x,y} L^q(x,y;u,s,\tau),$$

and the minimum is achieved. The proof is completed by finally recalling the definition of $\alpha^* = \alpha^*(\tau)$ and using Proposition 3. ∎

Particular Case. We consider the case where $Q(\cdot) = 1 \in R$. This means that among $\{x^T P(\cdot) \mid x \in R^n\}$ we have to find a function closest to f. Since this set is a finite-dimensional subspace, our problem has a solution. In this example we calculate our Lagrangian. Clearly, $B = (1,\dots,1)^T$, $y \in R$, $d \triangleq DB = (f(t_1),\dots,f(t_{n+1}))^T$, and denoting $v = \sum_{i=1}^{n+1} u_i$ our Lagrangian is

$$L^{n+1}(x,y;v,s,\tau) = \frac{(y-1)\sum_{i=1}^{n+1} u_i + y}{s^T\,|Ax-yd| + \eta \vee y} = \frac{(y-1)\,v + y}{s^T\,|Ax-yd| + \eta \vee y}. \tag{13}$$

Any saddle point of (13) satisfies $y \geqslant 1$, since otherwise, for any fixed x,

$$\min_{\substack{s\in\Delta^{n+1}, v\geqslant 0\\ \tau\in T^{n+1}}} L^{n+1}(x,y;v,s,\tau) = -\infty.$$

Moreover, the following inequality holds for each fixed $s \in \Delta^{n+1}$, $\tau \in T^{n+1}$, $v \geqslant 0$, $x \in R^n$, and $0 < y < 1$:

$$\begin{aligned} L^{n+1}(x,y;v,s,\tau) &\leqslant \frac{y}{s^T\,|Ax-yd| + \eta \vee y} \\ &= \frac{1}{s^T\,|Ax'-d| + (\eta \vee y)/y} \leqslant \frac{1}{s^T\,|Ax'-d| + 1} \\ &= L^{n+1}(x',1;v,s,\tau), \end{aligned}$$

where $x' \triangleq (1/y)\,x$, while for $y \leqslant 0$ we have

$$L^{n+1}(x,y;v,s,\tau) < 0 < L^{n+1}(x,1;v,s,\tau).$$

We conclude that the range of y can be restricted to $\{y \mid y \geqslant 1\}$ without affecting either the saddle value or the saddle-point set of (13). Thus, our minimax statement is

$$\min_{x:y\geqslant 1} \min_{\substack{s\in\Delta^{n+1}, v\geqslant 0\\ \tau\in T^{n+1}}} \frac{(y-1)\,v + y}{s^T\,|Ax-yd| + y}. \tag{14}$$

Furthermore, recalling our $\tilde{\mathbf{P}}(T)$ with $Q(\cdot) \equiv 1$, we deduce that if (x, y) is a solution of $\tilde{\mathbf{P}}(T)$, then for any $\lambda \geqslant 1$, $\lambda(x, y)$ is also a solution. Thus applying Theorem 2, we deduce that if $[x, y; v, s, \tau]$ is a saddle point of (14), then for any $\lambda \geqslant 1$, $[\lambda x, \lambda y; v, s, \tau]$ is also a saddle point. It follows that any saddle point of (14) satisfies $v = 0$. Let $x \in R^n$ and $y \geqslant 1$ be fixed. Then

$$\min_{\substack{s \in \Delta^{n+1}, v \geqslant 0 \\ \tau \in T^{n+1}}} \frac{(y-1)\, v + y}{s^T\, |Ax - yd| + y} = \min_{\substack{s \in \Delta^{n+1} \\ \tau \in T^{n+1}}} \frac{y}{s^T\, |Ax - yd| + y}.$$

Thus if we restrict the range of v to $v = 0$, this does not affect the other components of the saddle points of (14). Hence, our minimax problem is reduced to

$$\max_{x, y \geqslant 1} \min_{\substack{s \in \Delta^{n+1} \\ \tau \in T^{n+1}}} \frac{y}{s^T\, |Ax - yd| + y}.$$

Normalizing $y = 1$, we get

$$\inf \tilde{\mathbf{P}}(T) = \min_{x} \max_{\substack{s \in \Delta^{n+1} \\ \tau \in T^{n+1}}} s^T\, |Ax - d|.$$

By Theorem 3, its saddle points express all the solutions of $\tilde{\mathbf{P}}(T)$ with $Q(\cdot) \equiv 1$ and $y = 1$ and all the sets $T_0 \in \Sigma^{n+1}$.

This result can be obtained from the definition of α^* using saddle-point duality for convex programming (actually linear programming) and following the same technique as in Theorem 3.

We conclude this paper by noting that the reason for presenting a second saddle-point theorem is the piecewise differentiability of L^q and the differentiable property of L^ω. Suppose that $\inf \mathbf{P}(T) > 0$ and let $0 < \eta < 1$; denoting $\theta \triangleq (\theta_1, \ldots, \theta_\omega)$, the set

$$\Omega \triangleq \bigcap_{i=1}^{\omega} \left\{ (x, y, \theta) \;\middle|\; \left| f(\theta_i) - \frac{x^T P(\theta_i)}{y^T Q(\theta_i)} \right| > 0,\ y^T Q(\theta_i) > \eta \right\}$$

is open in $R^n \times R^m \times T^\omega$. Moreover, in view of Proposition 3 and the minimality of ω, it follows that for any $\tau \triangleq (t_1, \ldots, t_\omega)$ with $T_0 = \{t_1, \ldots, t_\omega\} \in \Sigma^\omega$ and any minimizing sequence (x_l, y_l) of $\tilde{\mathbf{P}}(T_0)$ we have

$$(x_l, y_l, \tau) \in \Omega \qquad \text{for} \quad l \text{ large enough;}$$

L^ω is differentiable in respect of x, y, u, and s at any point of the set $\operatorname{dom} L^\omega \cap \Omega$. Note that $\operatorname{dom} L^\omega \cap \Omega$ is an open set in $R^n \times R^m \times T^\omega$ and contains all the points $(x_l, y_l; u, s, \tau)$ satisfying:

(1) $\tau = (t_1, \ldots, t_\omega)$ with $T_0 \triangleq \{t_1, \ldots, t_\omega\} \in \Sigma^\omega$;

(2) (x_l, y_l) is a minimizing sequence of $\tilde{\mathbf{P}}(T_0)$ for any large enough l;

(3) $(x_l, y_l; u, s, \tau) \in \text{dom}\, L^\omega$.

In particular, if $\mathbf{P}(T)$ has a solution, dom $L^\omega \cap \Omega$ contains all the saddle points of L^ω. If, in addition to our supposition, T is included in an open set of a finite dimensional space and the functions f, P, and Q are differentiable on this open set, then L^ω is differentiable on dom $L^\omega \cap \Omega$. The augmented Lagrangian appearing in Section 3 of [6] can also be used, and in that case the above remark also applies.

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